K21P 4210

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT1C02 : Linear Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this part. **Each** question carries **4** marks.

1. Let T be a linear operator on R^3 defined by
$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$
What is the matrix of T in the standard ordered basis for R^4.

2. Let V be a finite dimensional vector space over the field F and let W be a subspace of V. Then prove that dim W + dim W^0 = dim V.

3. Find a 3 × 3 matrix for which the minimal polynomial is x^2.

4. Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T. Then prove that the minimal polynomial for T_W divides the minimal polynomial for T.

5. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Prove that T has no cyclic vector.

6. Apply the Gram-Schmidt process to the vectors $\beta_1 = (3, 0, 4)$, $\beta_2 = (-1, 0, 7)$, $\beta_3 = (2, 9, 11)$, to obtain an orthonormal basis for R^3 with standard inner product.

P.T.O.

PART – B

Answer **4** questions from this part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let T be a linear transformation from V into W. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

b) If S is any subset of a finite dimensional vector space V, then prove that $(S^{\circ})^{\circ}$ is a subspace spanned by S.

8. a) Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \ldots, \beta_n$ be any vectors in W. Then prove that there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$, $j = 1, \ldots, n$.

b) If A is an $m \times n$ matrix with entries in the field F, then prove that row rank (A) = column rank (A).

9. a) Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Then prove that the space L(V, W) is finite-dimensional and has dimension mn.

b) Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for C^3 defined by $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (2, 2, 0)$. Find the dual basis of B.

Unit – II

10. a) Let T be a linear operator on a finite-dimensional space V. Let $c_1, \ldots, c_k$ be the distinct characteristic values of T and let W_i be the null space of $(T - c_i I)$. Then prove that following are equivalent.

i) T is diagonalizable.

ii) The characteristic polynomial for T is $f = (x - c_1)^{d_1} \ldots (x - c_k)^{d_k}$ and $\dim W_i = d_i$, $i = 1, \ldots, k$.

iii) $\dim W_1 + \ldots + \dim W_k = \dim V$.

b) Prove that similar matrices have the same characteristic polynomial.

11. a) State and prove Cayley – Hamilton Theorem.

b) Let T be the linear operator on R^2, the matrix of which in the standard ordered basis is $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$. Prove that the only subspaces of R^2 invariant under T are R^2 and the zero subspace.

12. a) Let $\mathscr{F}$ be a commuting family of diagonalizable linear operators on the finite-dimensional vector space V. Prove that there exists an ordered basis for V such that every operator in $\mathscr{F}$ is represented in that basis by a diagonal matrix.

b) Find a projection E which projects R^2 onto the subspace spanned by (1, –1) along the subspace spanned by (1, 2).

Unit – III

13. a) State and prove primary decomposition theorem.

b) If V is the space of all polynomials of degree less than or equal to n over a field F, prove that the differentiation operator on V is nil potent.

14. a) Let F be a field and let B be an n × n matrix over F. Then prove that B is similar over the field F to one and only one matrix which is in rational form.

b) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Prove that T has no cyclic vector. What is the T-cyclic subspace generated by the vector (1, –1, 3) ?

c) Verify that the standard inner product on F^n is an inner product.

15. a) Let V be an inner product space and let $(\beta_1, \ldots, \beta_n\}$, be any independent vectors in V. Then construct orthogonal vectors $\alpha_1, \ldots, \alpha_n$ in V such that for each k = 1, 2, . . ., n the set $\{\alpha_1, \ldots, \alpha_k\}$ is a basis for the subspace spanned by $\beta_1, \ldots, \beta_k$.

b) Let V be a real or complex vector space with an inner product. Show that the quadratic form determined by the inner product satisfies the parallelogram law $|| \alpha + \beta ||^2 + || \alpha - \beta ||^2 = 2 || \alpha ||^2 + 2|| \beta ||^2$.